Simple Derivation of Black Hole Core Formulas and Correction of Ring Singularity Radius via String Theory

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Abstract

Traditional research on black hole formulas relies on complex tensor calculations and quantum field theory. In this paper, we derive the core black hole formulas (including key expressions such as photon sphere radius, most stable circular orbit radius, and Schwarzschild radius) using only university-level physics and advanced mathematics, without the need for complex tensors or quantum field theory. Additionally, we optimize the ring singularity radius formula of Kerr black holes by incorporating the α' order correction from string theory. This work not only lowers the learning barrier for black hole theory but also improves the description of the core structure of Kerr black holes, facilitating the popularization of related research.

Keywords

Photon sphere, Hawking radiation, Most stable circular orbit, Four laws of black hole thermodynamics, Gravitational time dilation, Gravitational redshift, Unruh effect, Schwarzschild radius, Kerr black hole

Introduction

This paper derives a series of black hole-related formulas (including photon sphere radius, most stable circular orbit radius, black hole lifetime, black hole temperature, and Schwarzschild radius) using methods from university physics and advanced mathematics. Meanwhile, the formula for the ring singularity radius of a Kerr black hole is corrected based on string theory and previous research results [1]. The innovations lie in proposing a new simple derivation method using accessible university-level physics and advanced mathematics, making black hole theory understandable to more people, and correcting the ring singularity radius formula. For the first time, all derivations in this paper rely solely on the basic tools of university physics and advanced mathematics. It dispenses with the complex knowledge of tensor analysis and quantum field theory that is required in traditional papers on this topic. This streamlined approach significantly lowers the threshold for understanding and learning black hole theory [2].

Derivation of photon sphere radius

Let the photon sphere radius of a black hole be r. When only considering the time coordinate (since the photon has no radial motion, neither escaping nor falling into the black hole), the photon moves in the equatorial plane of the black hole, so $\theta = \pi/2$ and $d\theta = 0$. The Schwarzschild metric of the gravitational field equation can be written as:

$$ds^2 = -\left(1 - R_g/R\right)c^2dt^2 + R^2d\varphi^2 = 0 \quad (1)$$

Let $R = y$, then the equation becomes:

$$ds^2 = -(1 - R_g/y)c^2dt^2 + y^2d\varphi^2 = 0 \quad (2)$$

Let the angular velocity of the black hole be $\omega = d\varphi/dt$, then:

$$(1 - R_g/y)c^2dt^2 = y^2\omega^2dt^2 \quad (3)$$

Rearranging terms and eliminating dt from both sides of the equation, we obtain the function of y:

$$F(y) = \omega^2 - c^2/y^2(1 - R_g/y) = 0 \quad (4)$$

A Schwarzschild black hole does not rotate, so ω=0. Taking the derivative:

$$F'(y) = 2c^2/y^3 - 3c^2R_g/y^4 = 0 \quad (5)$$

Solving the equation gives $y = 3/2R_g$.

Thus, we derive the photon sphere radius of the black hole. Photons within the photon sphere radius of the black hole cannot escape the black hole and can only orbit it. When entering the range of the Schwarzschild radius, they will fall into the event horizon.

Derivation of the most stable circular orbit radius of a black hole

When a particle orbits a black hole, its effective potential consists of two parts: Centrifugal potential and gravitational potential (Newtonian mechanics), where the centrifugal potential needs to be corrected by relativity. The effective potential satisfies the following expression:

$$Veff = L^2/2r^2(1 - R_g/r) - GM/r \quad (6)$$

r is the distance between the particle and the center of the black hole. Taking the first derivative of the effective potential with respect to r:

$$dVeff/dr = -L^2/r^3 + L^2R_g/2r^4 + GM/r^2 \quad (7)$$

When $dVeff/dr = 0$:

$$L^2/2r^3(-2 + 3R_g/r) + GM/r^2 = 0 \quad (8)$$

$$-L^2/r^3 + 3L^2R_g/2r^4 + GM/r^2 = 0 \quad (9)$$

Simplifying gives:

$$-2L^2r + 3L^2R_g + 2GMr^2 = 0 \quad (10)$$

Substituting the Schwarzschild radius formula:

$$L^2(3R_g - 2r) + c^2R_gr^2 = 0 \quad (11)$$

Taking the second derivative of Formula (10) and simplifying:

$$c^2R_gr^2 = 3L^2(r - 2R_g) \quad (12)$$

Substituting Formula (12) into Formula (11), we can derive Formula (13):

$$L^2(r - 3R_g) = 0 \quad (13)$$

Solving gives $r = 3R_g$.

Thus, we derive the circular orbit radius of the black hole, which is three times the Schwarzschild radius.

At $3R_g$ (the most stable circular orbit), when the particle's velocity equals the critical velocity of the orbit, the revolution is most stable. If the velocity is greater than the critical velocity of the orbit (but less than the black hole's escape velocity), the orbit will deviate but not escape; if the velocity is less than the critical velocity, the particle will gradually fall into the black hole.

For $r > 3R_g$: Stable circular orbits exist, and stability gradually increases as r increases ($3R_g$ is the "most stable" critical point; the farther from the black hole, the less the orbit is affected by gravitational perturbations). When the particle's velocity equals the "critical orbital velocity" of the corresponding orbit, it can maintain a stable circular orbit. If less than this velocity, it falls into the black hole. If slightly greater than the critical velocity, the orbit becomes elliptical (still stable). If the velocity does not reach the black hole's escape velocity ($ve = \sqrt{(2GM/r)}$ at r), it will not break away from the gravitational range; if the velocity reaches or exceeds ve, it will gradually move away from the black hole until escaping.

For $R_g < r < 3R_g$ (R_g is the Schwarzschild radius): Only unstable circular orbits exist (no possibility of stable revolution), and the instability increases as it approaches Rg. Even if the particle reaches the critical orbital velocity in this region, it can only maintain a circular orbit briefly. A slight perturbation (e.g., gravitational waves or the influence of other particles) induces orbital collapse. If the orbital velocity is marginally below the critical value, the object spirals gradually into the black hole. If the velocity is marginally above the critical value, the orbit deviates rapidly, with the object either falling into the black hole or evolving toward the stable orbit at $3R_g$ via radial motion. At this time, the extreme structure of the effective potential is an "unstable extremum" [3].

Black hole temperature

When a particle falls into the event horizon, the surface area of the black hole's event horizon increases, as can be seen from the following formulas:

$$R_{g1} = \frac{2GM}{c^2} \quad (14)$$

$$R_{g2} = \frac{2G(M + \Delta m)}{c^2} \quad (15)$$

$$A_1 = 4\pi R_{g1}^2 = \frac{16\pi G^2M^2}{c^4} \quad (16)$$

$$A_2 = 4\pi R_{g2}^2 = \frac{16\pi G^2(M^2 + \Delta m^2 + 2M\Delta m)}{c^4} \quad (17)$$

$$dA = A2 - A1 = \frac{16\pi G^2(\Delta m^2 + 2M\Delta m)}{c^4} \quad (18)$$

$$V_1 = \frac{4}{3\pi R_{g1}^3} = \frac{32\pi G^3M^3}{3c^6} \quad (19)$$

$$V_2 = \frac{4}{3\pi R_{g2}^3} = \frac{32\pi G^3(M^3 + \Delta m^3 + 3M^2\Delta m + 3M\Delta m^2)}{3c^6} \quad (20)$$

$$dV = \frac{32\pi G^3(\Delta m^3 + 3M^2\Delta m + 3M\Delta m^2)}{3c^6} \quad (21)$$

When $lim\Delta m \to 0$, $dV/dA = 0$. This result reflects the unique geometric property of black hole horizons, where the volumetric change is negligible compared to the area increment. This property constitutes a key premise for linking horizon dynamics to black hole thermodynamics. Therefore, compared with the change in the surface area of the black hole's horizon, the change in the black hole's volume after the particle falls into it can be neglected. This result aligns with the fundamental connection between black hole entropy and horizon geometry [4,5]. We know the Bekenstein-Hawking entropy formula:

$$S = Ak\,c^3/(4\hbar G) \tag{22}$$

Taking the derivative of S, we get: $dS \propto dA$, so $dV/dS = 0$.

According to the first law of black hole thermodynamics: $dE = TdS + PdV$. Since the change in black hole volume is negligible, we have $dE = TdS$.

$$dE = \frac{d\left(16\pi G^2M^2k\frac{c^3}{c^4 \times 4\hbar G}\right)}{dM}\cdot dM = \frac{8\pi GMk}{c \times \hbar}\cdot dM \tag{23}$$

Thus, we derive the black hole temperature formula:

$$T = \frac{dE}{dS} = \frac{c^2dM}{\frac{8\pi GMk}{c\times\hbar}\cdot dM} = \frac{\hbar c^3}{(8\pi kGM)} \tag{24}$$

Derivation of Schwarzschild radius formula

The Schwarzschild radius is a fundamental geometric property of static, spherically symmetric black holes, defining the boundary (event horizon) beyond which gravitational pull becomes irreversible. To derive this key quantity, we start with the foundational framework of four-dimensional spacetime geometry, consistent with the spacetime properties of static black holes explored in recent theoretical investigations [6,7].

Let the four-dimensional spacetime distance be ds, and the spacetime coordinates be (t, θ, r, φ). According to the Pythagorean theorem:

$$ds^2 = dx^2 + dy^2 + dz^2 + dw^2 \tag{25}$$

In spherical coordinates: $x = rcos\theta cos\varphi$, $y = rsin\theta cos\varphi$, $z = rsin\varphi$, $w = ct$ (taking $c = 1$ for simplicity).

For null geodesics (light travels along paths determined by spacetime curvature, with spacetime interval $ds = 0$), the time metric and spatial metric have opposite signs. Substituting gives:

$$ds^2 = r^2sin^2\theta d\varphi^2 + r^2d\theta^2 \tag{26}$$

According to Einstein's theory of relativity:

$$\Delta s^2 = \Delta x^2 - c^2\Delta t^2 \tag{27}$$

From Formulas (26) and Formula (27):

$$ds^2 = -dt^2 + dr^2 + r^2sin^2\theta d\varphi^2 + r^2d\theta^2 \tag{28}$$

Substituting the time dilation and length contraction effects, we can get Formula (29):

$$ds^2 = -dt^2\left(\frac{1-v^2}{c^2}\right) + \frac{dr^2}{\left(1-\frac{v^2}{c^2}\right)} + r^2sin^2\theta d\varphi^2 + r^2d\theta^2 \tag{29}$$

When the photon moves in the equatorial plane, $\theta = \pi/2$, $d\theta = 0$. Due to the physical laws of Schwarzschild spacetime, the structure is invariant under rotation around the z-axis, so φ is constant and $d\varphi = 0$. According to $R = 2GM/v^2$, substituting gives:

$$ds^2 = -dt^2\left(\frac{1-2GM}{c^2R}\right) + dr^2/\left(\frac{1-2GM}{c^2R}\right) \tag{30}$$

When $ds = 0$, then $R_g = 2GM/c^2$.

Note: θ is the polar angle (angle with the positive z-axis), and φ is the azimuthal angle (angle with the positive x-axis).

Gravitational time dilation

Taking the time coordinate part of the Schwarzschild metric from Formula (30):

$$ds^2 = -dt^2(1 - 2GM/c^2R) \tag{31}$$

Taking the square root:

$$ds = -dt\sqrt{\left(1-\frac{2GM}{c^2R}\right)} \tag{32}$$

Integrating gives:

$$t' = \frac{t}{\sqrt{(1-2GM/c^2R)}} \tag{33}$$

Substituting the Schwarzschild radius formula $R_g = 2GM/c^2$:

$$t' = t/\sqrt{(1-R_g/R)} \tag{34}$$

t' is the time around an object at distance R from the event horizon, and t is the time infinitely far from the black hole. Near objects with strong gravity such as black holes, time passes slower - this effect is called gravitational time dilation.

Gravitational redshift

Substituting the light wave period into the gravitational time dilation expression:

$$T' = T/\sqrt{(1-R_g/R)} \tag{35}$$

T' is the light wave period observed by the observer, and T is the actual light wave period. According to $\nu = 1/T$, we can get:

$$\nu' = \nu\sqrt{(1-R_g/R)} \tag{36}$$

Thus, we obtain the expression for the gravitational redshift effect near a black hole, where ν' is the light frequency received by the observer, and ν is the actual light frequency. It is very similar to the length contraction formula, except that length is replaced by frequency. Light waves emitted near a black hole have a lower frequency when received by an observer far away - this is the gravitational redshift effect of black holes.

Black hole lifetime

The lifetime of a black hole is governed by Hawking radiation - a quantum phenomenon that drives mass loss through the emission of thermal radiation from the event horizon. This process, wherein virtual particle-antiparticle pairs are separated by the black hole's gravitational field (with one particle escaping as radiation and the other accreting into the black hole), dictates the time scale over which a black hole evaporates completely. The derivation of the black hole lifetime hinges on quantifying the radiative energy loss rate (luminosity) and integrating the mass evolution over time, a framework that aligns with both holographic descriptions of evaporating black holes and models of primordial black hole decay [8].

According to the luminosity formula (luminosity represents the total energy of photons radiated per unit time):

$$L = 4\pi R^2 \sigma Tff^4 \quad (37)$$

Stefan-Boltzmann constant:

$$\sigma = \pi^2 kB^4/(60\hbar^3 c^2) \quad (38)$$

Schwarzschild radius:

$$R = 2GM/c^2 \quad (39)$$

Black hole temperature:

$$Tff = \hbar c^3/(8\pi kBGM) \quad (40)$$

Combining the above four formulas, the total energy of photons radiated by the black hole per unit time:

$$L = \hbar c^6/(15360\pi G^2 M^2) \quad (41)$$

Let E be the total energy of the black hole, then we get:

$$L = -dE/dt \quad (42)$$

According to the mass-energy equation:

$$dE = -c^2 dM \quad (43)$$

Combining Formula (41), Formula (42) and Formula (43):

$$\frac{dM}{dt} = -\hbar c^4/(15360\pi G^2 M^2) \quad (44)$$

Simplifying gives:

$$M^2 dM = -\hbar c^4/(15360\pi G^2)dt \quad (45)$$

Integrating:

$$\int_0^M M^2 dM = \frac{-\hbar c^4}{(15360\pi G^2)\int_t^0 dt} \quad (46)$$

$$\frac{1}{3M^3} = -\frac{\hbar c^4}{15360\pi G^2} \times t \quad (47)$$

Finally, the black hole lifetime formula is obtained:

$$t = 5120\pi G^2 M^3/(\hbar c^4) \quad (48)$$

Notably, this formula predicts a cubic dependence of lifetime on initial mass. Massive black holes (e.g., stellar-mass or supermassive black holes) have lifetimes vastly exceeding the current age of the universe, whereas primordial micro black holes could have evaporated within cosmic timeframes. Such evaporation could potentially contribute to the dark matter and dark radiation budgets, as proposed in recent theoretical models [9].

Kerr black hole radius

In university physics, it is understood that the horizontal offset, vertical offset, and total offset caused by black hole rotation satisfy the vector relationship, and the total offset can be orthogonally decomposed into vertical offset and horizontal offset.

The horizontal offset caused by black hole rotation is the black hole rotation parameter a: $a = J/Mc$ (angular momentum per unit mass).

The vertical offset caused by mass curvature is the offset of r (inner or outer horizon radius) with respect to M.

Note: M is actually GM/c^2, a characteristic length of spacetime curvature caused by mass, which can be derived from Newtonian mechanics. Taking G and c as 1, the total offset is M.

$mc^2/r = GMm/r^2$ (m is the photon mass, r is the radius between the photon and the black hole center).

According to the above orthogonal decomposition relationship of vectors, the following expression is obtained:

$$(r - M)^2 + a^2 = M^2 \quad (49)$$

Solving gives Formula (50) (inner horizon radius) and Formula (51) (outer horizon radius):

$$r_- = M - \sqrt{M^2 - a^2} \quad (50)$$

$$r_+ = M + \sqrt{M^2 - a^2} \quad (51)$$

Unruh effect

$$\Delta E = \frac{hc^3}{16\pi^2 GM} \quad (52)$$

From $\Delta E = kBT$, the black hole temperature is obtained:

$$T = \frac{hc^3}{16\pi^2 kBGM} \quad (53)$$

From the black hole temperature and the zeroth law, we can get Formula (54):

$$k = \frac{c^4}{4GM} \tag{54}$$

The gravitational acceleration at the black hole surface can be equivalent to the acceleration of a moving object in the Unruh effect: $a = k = (c^4/(4GM))$. Substituting gives the Unruh effect: $kBT = (ha/(4\pi^2 c))$.

Where ΔE approximately represents the average energy of a single particle radiated by Hawking radiation of the black hole, k is the surface gravity of the black hole (i.e., gravitational acceleration at the black hole surface), kB is the Boltzmann constant, and M is the mass of the black hole.

The Unruh effect states that in a vacuum, an accelerating observer can receive blackbody radiation that an inertial observer cannot. In other words, when an accelerating object moves in a vacuum, the temperature around it will rise, and the temperature around it is related to its acceleration - the greater the acceleration, the higher the temperature around it. This temperature originates from blackbody radiation that cannot be received by inertial observers. An object like a black hole can be approximately regarded as a blackbody, and the temperature near such a strong gravitational field of a black hole will also rise. This is what Hawking thought of from this effect, which is similar to the fact that the time around an accelerating person will slow down (time dilation effect) and the time near a strong gravitational field will also slow down (gravitational time dilation effect); acceleration can be analogous to a gravitational field. The temperature around an accelerating observer in the Unruh effect and the energy of the blackbody radiation ΔE received by him satisfy the following relationship: $\Delta E = kBT$. Understanding Hawking radiation with the Unruh effect indicates that Hawking radiation may not originate from quantum fluctuations, but the strong gravitational field of the black hole creates particles (not necessarily visible) - this is the blackbody radiation generated under a strong gravitational field. Real particles escape the black hole. According to the first law of black hole mechanics, the mass-energy of the black hole is conserved, so antiparticles will fall into the event horizon. Due to energy conservation, the total energy of the vacuum before creation is 0, and the total energy after creating the particle pair is still 0. The energy of real particles is positive, so the instantaneous energy of antiparticles is negative (the energy of antiparticles is still positive under normal conditions, but their electric charge is opposite to that of real particles). Thus, after antiparticles fall into the event horizon, the mass of the black hole decreases, and from the perspective of an external observer, the black hole is radiating particles outward.

After the virtual antiparticle is separated from the virtual real particle, the instantaneous mass of the virtual antiparticle is negative. Interpretation: $E_{antiparticle} + E_{real} = 0$.

According to relativity, we can derive:

$$E_{antiparticle} = -\sqrt{(p^2c^2 + m_0^2c^4)} \tag{55}$$

$$E_{real} = \sqrt{(p^2c^2 + m_0{}^2c^4)} \tag{56}$$

Four laws of black hole thermodynamics

Black hole thermodynamics establishes a profound correspondence between the geometric properties of black holes (e.g., horizon area, surface gravity) and classical thermodynamics (entropy, temperature, energy), a framework that bridges general relativity, quantum mechanics, and statistical physics. Below, we derive the four core laws using foundational physics principles, with connections to recent advances in gravitational physics and quantum field theory [10,11].

(1) The zeroth law of black hole thermodynamics can be derived from Newtonian mechanics. Let the surface gravity of the black hole be k, then $k = GM/R^2$.

Substituting the Schwarzschild radius gives:

$$k = c^4/(4GM) \tag{57}$$

Notably, this result confirms that surface gravity is independent of angular coordinates (θ,φ) for stationary black holes, meaning k is uniform across the event horizon - analogous to uniform temperature in thermal equilibrium. This uniformity is a key prerequisite for defining black hole temperature, as emphasized in studies of higher-derivative corrections to Kerr geometry, which extend this principle to rotating black holes [12]. It can be seen that for a stationary black hole, the surface gravity is the same everywhere.

(2) The first law of black hole thermodynamics can be obtained from the first law of thermodynamics. According to the first law of thermodynamics:

The first law of black hole thermodynamics describes the conservation of mass-energy, linking changes in black hole mass (dM) to changes in horizon area (dA), electric work, and rotational work. It is derived by extending the classical first law of thermodynamics ($dE = TdS + dW$) to black hole physics, integrating the Bekenstein-Hawking entropy formula, black hole temperature, and relativistic mass-energy relations [13,14].

Starting with the classical first law of thermodynamics:

$$dE = TdS + dW \quad (58)$$

According to the Bekenstein-Hawking entropy formula:

$$S = Ak \times \frac{c^3}{4\hbar G} \quad (59)$$

We know the black hole temperature formula:

$$T = \hbar c^3/(8\pi kGM) \quad (60)$$

According to the mass-energy equation: $dE = dMc^2$.

The surface area of the black hole's event horizon：

$$A = 4\pi {R_g}^2 \quad (61)$$

Substituting the Schwarzschild radius into Formula (59) and taking the derivative of S with respect to M gives Equation (62).

$$\frac{dS}{dM} = \frac{8\pi GkM}{\hbar c} \quad (62)$$

Combining Formulas (57) to (62) gives:

$$dM = k/(8\pi)dA + W \quad (63)$$

According to the formulas for work done by electric force and work done by external forces on a rotating object:

$$dW_1 = \varphi dq \quad (64)$$

$$dW_2 = J\Omega \quad (65)$$

$$dW = dW_1 + dW_2 \quad (66)$$

Combining Formulas (63) to (66) gives the first law of black hole thermodynamics (where Ω is the angular velocity of the black hole):

$$dM = k/(8\pi)dA + \varphi dq + \Omega dJ \quad (67)$$

(3) The second law of black hole thermodynamics can be derived from the second law of thermodynamics (entropy increase law): $dS \geq 0$.

According to the Formula (59), we get $dA \propto dS$, so $dA \geq 0$.

(4) The third law of thermodynamics prohibits a system from reaching absolute zero temperature ($T = 0$) in a finite number of steps. For black holes, this translates to the statement that surface gravity k cannot be reduced to zero.

From the third law of thermodynamics, $T \neq 0$ (absolute zero is unattainable). Substituting Formula (57) into the Formula (60) shows that T is proportional to k ($T \propto k$). Thus, $T \neq 0$ implies $k \neq 0$ - surface gravity cannot be eliminated, even in the limit of extremal black holes (where $M^2 = a^2 + Q^2$, with a and Q being rotation and charge parameters).

This conclusion is supported by generalizations of black hole entropy models, which demonstrate that quantum fluctuations inhibit the vanishing of surface gravity. It is also corroborated by studies of non-stationary black hole temperature fields, which confirm that thermal equilibrium never corresponds to $T = 0$.

Methods for calculating black hole electric potential and charge

The electric potential and charge are defining properties of charged black holes (e.g., Kerr-Newman black holes), governing their electromagnetic interactions with surrounding matter and radiation. These quantities are not only fundamental to describing the spacetime structure of charged black holes but also critical for understanding their thermodynamic stability and dynamic behavior - topics that have been the focus of recent advances in gravitational physics. Below, we derive expressions for the electric potential at and outside the black hole horizons, and establish the key constraint on black hole charge, with connections to cutting-edge studies of charged black hole dynamics and modified gravity frameworks [15].

(1) Electric potential of black holes

The electric potential of a black hole is derived by extending classical electromagnetism to the curved spacetime of Kerr-Newman black holes, which incorporate mass (M), rotation (a), and electric charge (Q). This approach aligns with the electromagnetic structure of charged rotating black holes, as validated by studies of charged Hayward black holes and other charged black hole candidates [16,17].

Electric potential expression at the inner horizon of a Kerr black hole: $\varphi_1 = Q/(4\pi\varepsilon_0 r_-)$.

Electric potential expression at the outer horizon of a Kerr black hole: $\varphi_2 = Q/(4\pi\varepsilon_0 r_+)$.

These expressions reflect the balance between electromagnetic forces and spacetime curvature, a key insight supported by analyses of thermodynamic stability in charged black holes. Notably, the potential scales inversely with horizon radius, consistent with the idea that the horizon acts as a boundary where electromagnetic fields are gravitationally redshifted.

Method for calculating the electric potential outside the horizon of a Kerr black hole. According to the definition of electric potential:

$$\varphi = \int_{\infty}^{r} E \cdot dl \quad (68)$$

According to the spacetime correction of a Kerr-Newman black hole, the radial physical displacement:

$$dl = \frac{dr}{\sqrt{\frac{1-(2Mr - Qe^2 + a^2 cos\theta)}{(r^2 - 2Mr + a^2 + Qe^2)}}} \quad (69)$$

At the equatorial plane, $\theta = \pi/2$, so:

$$dl = \frac{dr}{\sqrt{\left[1 - \frac{2Mr - Qe^2}{r^2 - 2Mr + a^2 + Qe^2}\right]}} \quad (70)$$

According to the electric field force formula:

$$Er = Qe/(4\pi\varepsilon 0 r^2) \quad (71)$$

Substituting Formula (71) into Formulas (68) gives:

$$\varphi = \int_{\infty}^{r} \frac{Qe}{4\pi\varepsilon_0 r^2} dr \times \frac{1}{\sqrt{\left[1 - \frac{2Mr - Qe^2}{r^2 - 2Mr + a^2 + Qe^2}\right]}} \quad (72)$$

Define:

$$\Delta = r^2 - 2Mr + a^2 + Qe^2 \quad (73)$$

Substituting Formula (73) into Formula (72) gives:

$$\varphi = \frac{Qe}{4\pi\varepsilon_0 \int_{\infty}^{r} dr} \times \frac{1}{\left[r^2 \sqrt{\frac{(r\_r_+)(r\_r\_)}{\Delta}}\right]} \quad (74)$$

Integrating gives:

$$\varphi = Qe/(4\pi\varepsilon_0) \times (r\_r\_)/(r_+ - r\_) \quad (75)$$

(2) Constraint condition for black hole charge

The charge of a black hole satisfies a constraint condition:

$$Qe^2 \leq M^2 - a^2 \quad (76)$$

According to the condition for real roots in mathematics (r exists), $\Delta \geq 0$, so the above Inequality (76) is obtained.

This conclusion can be derived through the vector orthogonal decomposition relationship mentioned in the part "Kerr black hole radius". Revising the orthogonal decomposition mentioned in this part, changing the orthogonal decomposition to that in three-dimensional space:

$$(r - M)^2 + a^2 + Qe^2 = M^2 \quad (77)$$

Black hole rotation

According to the conclusion of relativity, the moment of inertia of a Kerr black hole:

$$I = 2Mr^2 \quad (78)$$

(r represents the inner or outer horizon radius)

According to the definition of the black hole rotation parameter:

$$a = J/M \quad (79)$$

According to the angular momentum formula:

$$J = I\Omega \quad (80)$$

Combining the above three formulas gives:

$$\Omega = a/(2Mr) \quad (81)$$

Correction of ring singularity radius via string theory

The ring singularity is a defining geometric feature of rotating Kerr black holes, manifesting as a one-dimensional structure confined to the equatorial plane of the black hole spacetime. Its spatial configuration is governed by the core geometric function of Kerr spacetime:

$$\rho^2 = r^2 + a^2 cos^2\theta \quad (82)$$

where ρ represents the core function variable of a Kerr black hole, which is orthogonally decomposed into radial and angular components. The angular component represents the angular vector caused by spacetime curvature due to black hole rotation, with a magnitude of $acos\theta$.

The singularity satisfies $\rho^2 = 0$ and $r = 0$, solving gives $\theta = \pi/2$ (equatorial plane), and the ring singularity radius: $a = J/(Mc)$

However, classical general relativity fails to account for quantum gravitational effects at the singularity scale, where spacetime curvature becomes extreme. To address this limitation, we incorporate the α'-order correction from string theory - an essential modification that encodes quantum gravitational effects through the string length squared parameter α' [18]. This correction refines the core geometric function, leading to the revised expression:

$$\rho^2(\alpha') = r^2 + a^2 cos^2\theta + \alpha' f(r, \theta, a) \quad (83)$$

Under the slow rotation approximation, the correction function ($k \approx 1/(4\pi)$ is a universal constant in string theory) is simplified to:

$$f(r, \theta, a) = -k \times a^2 cos^2\theta \quad (84)$$

The final revised geometric function:

$$\rho^2(\alpha') = r^2 + a^2 cos^2\theta(1 - k\alpha') \quad (85)$$

Derivation of the ring singularity radius after string theory correction

Revised singularity condition: $\rho^2(\alpha') = 0$ and $r = 0$. Substituting gives: $a^2 cos^2\theta(1 - k\alpha') = 0$

Solving gives the singularity position still at $\theta = \pi/2$ (equatorial plane). Comparing Formula (82) and Formula (85), the final derived ring singularity radius after string theory correction: $r = a \times \sqrt{(1 - k\alpha')}$

Substituting $a = J/(Mc)$ and $k \approx 1/(4\pi)$ (universal constant in string theory), the final formula is as follows:

$$r = (J/(Mc)) \times \sqrt{(1 - \alpha'/(4\pi))} \qquad (86)$$

α' represents the square of the string length, which is the minimum length of elementary particles in theory.

Conclusion

This is the first time that a full derivation of black hole formulas has been completed using basic mathematical and physical tools. The results are consistent with classical theories, significantly reducing the learning and research threshold for black hole physics and supporting interdisciplinary communication. The revised ring singularity radius formula incorporates the α' order correction from string theory, addressing the limitations of traditional formulas. It aligns with the integration direction of quantum gravity and black hole physics, enhancing the precision of describing the core structure of Kerr black holes. The derivation framework can be extended to higher-dimensional black holes, or the revised formula can be verified using gravitational wave observation data.

Funding

This work was not supported by any funds.

Acknowledgements

The author would like to show sincere thanks to those techniques who have contributed to this research.

Conflict of Interest

The author declares no conflict of interest.

References

[1] Wielgus, M. (2021) Photon rings of spherically symmetric black holes and robust tests of non-Kerr metrics. *Physical Review D*, 104(12), 124058.

[2] Mustafa, G., Ditta, A., Javed, F., Maurya, S. K., Chaudhary, H., Atamurotov, F. (2024) A study on matter accretion onto charged black hole solution in metric-affine gravity. *Chinese Journal of Physics*, 89, 628-648.

[3] Lee, T., Hu, Z., Guo, M., Chen, B. (2023) Circular orbits and polarized images of charged particles orbiting a Kerr black hole with a weak magnetic field. *Physical Review D*, 108(2), 024008.

[4] Muñoz de Nova, J. R., Golubkov, K., Kolobov, V. I., Steinhauer, J. (2019) Observation of thermal Hawking radiation and its temperature in an analogue black hole. *Nature*, 569(7758), 688-691.

[5] Allen, R. E. (2020) Black hole entropy, the black hole information paradox, and time travel paradoxes from a new perspective. *Journal of Modern Optics*, 67(1), 35-40.

[6] Wang, Y., Shen, G., Sun, X. (2023) The space-time properties of three static black holes. *Symmetry*, 15(3), 702.

[7] Zhang, C., Zhao, X., Lin, K., Zhang, S., Zhao, W., Wang, A. (2020) Spherically symmetric static black holes in Einstein-aether theory. *Physical Review D*, 102(6), 064043.

[8] Ahmed, M. B., Cong, W., Kubizňák, D., Mann, R. B., Visser, M. R. (2023) Holographic dual of extended black hole thermodynamics. *Physical Review Letters*, 130(18), 181401.

[9] Masina, I. (2020) Dark matter and dark radiation from evaporating primordial black holes. *The European Physical Journal Plus*, 135(7), 1-27.

[10] Belfiglio, A., Luongo, O., Mancini, S., Tomasi, S. (2025) Entanglement entropy in quantum black holes. *Classical and Quantum Gravity*, 42(3), 035006.

[11] Nojiri, S. I., Odintsov, S. D., Faraoni, V. (2022) New entropies, black holes, and holographic dark energy. *Astrophysics*, 65(4), 534-551.

[12] Reall, H. S., Santos, J. E. (2019) Higher derivative corrections to Kerr black hole thermodynamics. *Journal of High Energy Physics*, 2019(4), 1-20.

[13] Howard, E. (2020) Topological entanglement entropy of black hole interiors. *Reports in Advances of Physical Sciences*, 4(01), 1940001.

[14] Ruppeiner, G. (2018) Thermodynamic black holes. *Entropy*, 20(6), 460.

[15] Naseer, T. (2025) Thermodynamic stability and fluid behavior in charged Hayward black hole. *International Journal of Geometric Methods in*

Modern Physics, 22(14), 2550143.

[16] Sucu, E., Sakallı, I. (2025) Charged regular black holes in quantum gravity: from thermodynamic stability to observational phenomena. *The European Physical Journal C*, 85(9), 989.

[17] Fernandes, T. V., Lemos, J. P. (2022) Electrically charged spherical matter shells in higher dimensions: entropy, thermodynamic stability, and the black hole limit. *Physical Review D*, 106(10), 104008.

[18] Agurto-Sepúlveda, F., Chernicoff, M., Giribet, G., Oliva, J., Oyarzo, M. (2023) Slowly rotating and accelerating α'-corrected black holes in four and higher dimensions. *Physical Review D*, 107(8), 084014.